FOURIER-MUKAI TRANSFORM ON ABELIAN SURFACES

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0. Introduction

Let (X,H) be a pair of an abelian surface X and an ample line bundle H on X. Let $\langle\ ,\ \rangle$ be a bilinear form on $H^{ev}(X,\mathbb{Z}) := \bigoplus_i H^{2i}(X,\mathbb{Z})$ defined by

$$\langle x,y\rangle := \int_X (x_1 \cup y_1 - x_0 \cup y_2 - x_2 \cup y_0) \tag{0.1}$$

where $x=(x_0,x_1,x_2), y=(y_0,y_1,y_2)$ with $x_i,y_i \in H^{2i}(X,\mathbb{Z})$. For an object $E \in \mathbf{D}(X)$, we define the Mukai vector $v(E) \in H^{ev}(X,\mathbb{Z})$ of E as the Chern character of E. We also call an element $v \in H^*(X,\mathbb{Z})$ Mukai vector, if $v=v(E)$ for an object $E \in \mathbf{D}(X)$.

We denote the coarse moduli space of S-equivalence classes of semi-stable sheaves E with $v(E)=v$ by $\overline{M}_H(v)$ and the open subscheme consisting of stable sheaves by $M_H(v)$. We also denote the moduli stack of semi-stable sheaves by $\mathcal{M}_H(v)^{ss}$. Let $Y := M_H(v_0)$ be the moduli space of stable semi-homogeneous sheaves on X. Assume that Y is a fine moduli space, that is, there is a universal family $\mathbf{E}$ on $Y \times X$. We define the integral functor $\Phi^{\mathbf{E}}_{Y\to X}$ as

$$\begin{array}{ccc} \mathbf{D}(Y) & \to & \mathbf{D}(X) \\ y & \mapsto & \mathbf{R}p_{X*}(\mathbf{E} \otimes p_Y^*(y)), \end{array} \tag{0.2}$$

where $p_X : X \times Y \to X$ (resp. $p_Y : X \times Y \to Y$) be the projection. Let $\mathbf{D}(X)_{op}$ be the opposite category of $\mathbf{D}(X)$ and define the equivalence

$$\begin{array}{cccc} D: & \mathbf{D}(X) & \to & \mathbf{D}(X)_{op} \\ & x & \mapsto & x^{\vee} = \mathbf{R}\mathcal{H}om(x,\mathcal{O}_X). \end{array} \tag{0.3}$$

Definition 0.1. We call equivalences $\mathbf{D}(Y) \to \mathbf{D}(X)$ and $\mathbf{D}(Y) \to \mathbf{D}(X)_{op}$ the *Fourier-Mukai transform.*

$\Psi^{\mathbf{E}}_{Y\to X} : H^*(Y,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ denotes the cohomological transform induced by $\mathbf{E}$.

Theorem 0.1. *Let $w \in H^*(Y,\mathbb{Z})$ be a primitive Mukai vector with $\langle w^2\rangle > 0$. Let H' be an ample divisor on Y witch is general with respect to w. We set $v = \Psi^{\mathbf{E}}_{Y\to X}(w)$. We assume that H is general with respect to v. Then there is an autoequivalence $\Phi^{\mathbf{F}}_{X\to X} : \mathbf{D}(X) \to \mathbf{D}(X)$ such that for a general $E \in M_{H'}(v)$, $F := \Phi^{\mathbf{F}}_{X\to X} \circ \Phi^{\mathbf{E}}_{Y\to X}(E)$ is a stable sheaf with $v(F)=v$ or $F^{\vee}$ is a stable sheaf with $v(F^{\vee})=v$. In particular, $M_{H'}(w)$ is birationally equivalent to $M_H(v)$.*

Since the moduli space of semi-stable sheaves is irreducible, the same assertion in Theorem 0.1 also holds for a general stable sheaf E with a non-primitive vector. This is a partial generalization of a result in [Y4], which is conjectured in [Y2, Conj. 4.16]. If X is a $K3$ surface, then a similar result is conjectured by Tom Bridgeland. In particular, the idea of replacing $\Phi^{\mathbf{E}}_{Y\to X}(E)$ by another Fourier-Mukai transform $\Phi^{\mathbf{F}}_{X\to X} \circ \Phi^{\mathbf{E}}_{Y\to X}(E)$ is due to him.

1. Preliminaries

1.1. **A family of 2-extensions.** In this section, we recall or prepare some necessary results to prove Theorem 0.1. We start with a possibly well-known result on a family of 2-extensions.

Definition 1.1. Let

$$\mathcal{V}_\bullet : \cdots \to \mathcal{V}_{-1} \to \mathcal{V}_0 \to \mathcal{V}_1 \to \cdots \tag{1.1}$$

be a complex on $X \times T$. $\mathcal{V}_\bullet$ is flat, if $\mathcal{V}_i$ are flat over T.

We shall construct a family of 2-extensions:

$$0 \to A_0 \to V_0 \to V_1 \to A_1 \to 0. \tag{1.2}$$

Let v_0, v_1 be Mukai vectors of coherent sheaves on X. Let Q_i, $i=0,1$ be the open subscheme of the quot-scheme $\mathrm{Quot}^{v_i}_{W_i\otimes\mathcal{O}_X(-n_i)/X}$ parametrizing all quotients $W_i \otimes \mathcal{O}_X(-n_i) \to A_i$ with $v(A_i)=v_i$ such that $W_i \cong H^0(X,A_i(n_i))$ and $H^j(X,A_i(n_i))=0, j>0$. Let $\mathcal{A}_i$ be the universal quotient and $\mathcal{I}_i$ the universal

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subsheaf. We take an integer m such that (i) $R^j p_{Q_1*}(\mathcal{I}_1(m)) = 0, j > 0$, (ii) $\mathcal{U} := p_{Q_1*}(\mathcal{K}_1(m))$ is locally free and (iii) $\psi_0 : p_{Q_1}^*(\mathcal{U}) \to \mathcal{I}_1(m)$ is surjective. We set $\mathcal{J} := \ker(\psi_0)(-m)$. Let $\widetilde{Q}_1 \to Q_1$ be the principal GL-bundle associated to $\mathcal{U}$. Then we have a trivialization $\mathcal{U} \cong U \otimes \mathcal{O}_{\widetilde{Q}_1}$, where U is a vector space. Let $\widetilde{\mathcal{I}}_i$ (resp. $\widetilde{\mathcal{J}}, \widetilde{\mathcal{A}}_i$) be the pull-back of $\mathcal{I}_i$ (resp. $\mathcal{J}, \mathcal{A}_i$) to $Q_0 \times \widetilde{Q}_1 \times X$. Then we have exact sequences

$$0 \to \widetilde{\mathcal{J}} \to U \otimes \mathcal{O}_{Q_0 \times \widetilde{Q}_1 \times X}(-m) \to \widetilde{\mathcal{I}}_1 \to 0, \tag{1.3}$$

$$0 \to \widetilde{\mathcal{I}}_i \to W_i \otimes \mathcal{O}_{Q_0 \times \widetilde{Q}_1 \times X}(-n_i) \to \widetilde{\mathcal{A}}_i \to 0. \tag{1.4}$$

If m is sufficiently large, then $\mathrm{Ext}^j_{p_{Q_0 \times Q_1}}(\widetilde{\mathcal{J}}, \widetilde{\mathcal{A}}_0) = 0$ and $\mathbb{E} := \mathrm{Hom}_{p_{Q_0 \times Q_1}}(\widetilde{\mathcal{J}}, \widetilde{\mathcal{A}}_0)$ is locally free. We have an exact sequence:

$$0 \to \mathrm{Hom}_{p_{Q_0 \times \widetilde{Q}_1}}(\widetilde{\mathcal{I}}_1, \widetilde{\mathcal{A}}_0) \to \mathrm{Hom}_{p_{Q_0 \times \widetilde{Q}_1}}(U \otimes \mathcal{O}_{Q_0 \times \widetilde{Q}_1 \times X}(-m), \widetilde{\mathcal{A}}_0) \to \mathbb{E} \to \mathrm{Ext}^1_{p_{Q_0 \times \widetilde{Q}_1}}(\widetilde{\mathcal{I}}_1, \widetilde{\mathcal{A}}_0) \to 0. \tag{1.5}$$

Let $\pi : P \to Q_0 \times \widetilde{Q}_1$ be the associated vector bundle of $\mathbb{E}$. Then we have a family of extensions

$$0 \to (\pi \times 1_X)^*(\widetilde{\mathcal{A}}_0) \to \widehat{\mathcal{V}}_0 \to (\pi \times 1_X)^*(\widetilde{\mathcal{I}}_1) \to 0. \tag{1.6}$$

We set $\widehat{\mathcal{V}}_1 := W_1 \otimes \mathcal{O}_{P \times X}(-n_1)$. Then we have a family of complexes

$$\widehat{\mathcal{V}}_\bullet : \widehat{\mathcal{V}}_0 \to \widehat{\mathcal{V}}_1 \tag{1.7}$$

such that $\widehat{\mathcal{V}}_i$ are flat over P, $H^j(\widehat{\mathcal{V}}_\bullet)$ are flat over P and $H^j(\widehat{\mathcal{V}}_\bullet)_x = (\mathcal{A}_j)_{\pi(x)}$.

Let S_i be a bounded set of coherent sheaves E_i on X with the Mukai vector v_i.

Proposition 1.1. *Let $\mathcal{V}_\bullet$ be a T-flat family of complexes on X parametrized by T such that $H^i(\mathcal{V}_\bullet)$ are flat families of coherent sheaves belonging to S_i. Then for any point $t \in T$, there is a neighborhood T_0 of t with the following prpperty: there is a quasi-isomorphism $\mathcal{V}'_\bullet \to \mathcal{V}_\bullet$ and a morphism $f : T \to P$ such that $f^*(\widehat{\mathcal{V}}_\bullet)$ is quasi-isomorphic to $\mathcal{V}'_\bullet$.*

Proof. Construction of $\mathcal{V}'_\bullet \to \mathcal{V}_\bullet$: Let $\mathcal{V}_\bullet := (\mathcal{V}_0 \xrightarrow{\phi} \mathcal{V}_1)$ be a flat family of complexes on $X \times T$ such that $H^i(\mathcal{V}_\bullet)$ are flat over T. Let $\mathcal{B}$ be the kernel of $\mathcal{V}_1 \to H^1(\mathcal{V}_\bullet)$. For a sufficiently large n, $R^j p_{T*}(\mathcal{B}(n)) = R^j p_{T*}(\mathcal{V}_1(n)) = R^j p_{T*}(H^1(\mathcal{V}_\bullet)(n)) = 0$ for $j > 0$ and we have an exact and commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & p_T^*(p_{T*}(\mathcal{B}(n))) \arrow[r] \arrow[d] & p_T^*(p_{T*}(\mathcal{V}_1(n))) \arrow[r] \arrow[d] & p_T^*(p_{T*}(H^1(\mathcal{V}_\bullet)(n))) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \mathcal{B}(n) \arrow[r] \arrow[d] & \mathcal{V}_1(n) \arrow[r] \arrow[d] & H^1(\mathcal{V}_\bullet)(n) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd} \tag{1.8}$$

By shrinking T if necessary,, we may assume that there is a lifting $\widetilde{\psi} : p_T^*(p_{T*}(H^1(\mathcal{V}_\bullet)(n))) \to \mathcal{V}_1(n)$ of ψ. We set $\mathcal{V}'_1 := p_T^*(p_{T*}(H^1(\mathcal{V}_\bullet)(n)))(-n)$ and set $\mathcal{K}_1 := \ker(\psi)(-n)$. Then we have a homomorphism $\mathcal{K}_1 \to \mathcal{B}$. Let $\mathcal{V}'_0$ be a coherent sheaf fitting in the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{V}_\bullet) \arrow[r] \arrow[d, equal] & \mathcal{V}'_0 \arrow[r] \arrow[d] & \mathcal{K}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(\mathcal{V}_\bullet) \arrow[r] & \mathcal{V}_0 \arrow[r] & \mathcal{B} \arrow[r] & 0.
\end{tikzcd} \tag{1.9}$$

Then $\mathcal{V}'_\bullet : \mathcal{V}'_0 \to \mathcal{V}'_1$ is quasi-isomorphic to $\mathcal{V}_\bullet$. We shall show that there is a local morphism $f : T \to P$ with a quasi-isomorphism $\mathcal{V}'_\bullet \to (f \times 1_X)^*(\widehat{\mathcal{V}}_\bullet)$ for a sufficiently large n.

Construction of $f : T \to P$: We take a trivialization $p_{T*}(H^i(\mathcal{V}_\bullet)(n_i)) \cong W_i \otimes \mathcal{O}_T$. Then we have a morphism $h : T \to Q_0 \times Q_1$ such that $(h \times 1_X)^*(\mathcal{A}_i) = H^i(\mathcal{V}_\bullet)$ as quotients of $W_i \otimes \mathcal{O}_{T \times X}(-n_i)$. If n is sufficiently large, then $\mathrm{Hom}(\mathcal{V}'_1, W_1 \otimes \mathcal{O}_{T \times X}(-n_1)) \to \mathrm{Hom}(\mathcal{V}'_1, H^1(\mathcal{V}_\bullet))$ is surjective. Hence there is a homomorphism $\mathcal{V}'_1 \to W_1 \otimes \mathcal{O}_{T \times X}(-n_1)$ and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_1 \arrow[r] \arrow[d] & \mathcal{V}'_1 \arrow[r] \arrow[d] & H^1(\mathcal{V}_\bullet) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (h \times 1_X)^*(\mathcal{I}_1) \arrow[r] & W_1 \otimes \mathcal{O}_{T \times X}(-n_1) \arrow[r] & H^1(\mathcal{V}_\bullet) \arrow[r] & 0,
\end{tikzcd} \tag{1.10}$$

where $\mathcal{I}_1$ means the pull-back of $\mathcal{I}_1$ to $Q_0 \times Q_1 \times X$. By our choice of n_i and n, we have

$$\begin{array}{ccc} \operatorname{Ext}^1_{p_T}(\mathcal{K}_1, H^0(\mathcal{V}_\bullet)) & \cong & \operatorname{Ext}^2_{p_T}(H^1(\mathcal{V}_\bullet), H^0(\mathcal{V}_\bullet)) \\ \uparrow & & \| \\ \operatorname{Ext}^1_{p_T}((h \times 1_X)^*(\mathcal{I}_1), H^0(\mathcal{V}_\bullet)) & \cong & \operatorname{Ext}^2_{p_T}(H^1(\mathcal{V}_\bullet), H^0(\mathcal{V}_\bullet)). \end{array} \tag{1.11}$$

Shrinking T if necessary, there is a coherent sheaf $\widetilde{\mathcal{V}}_0$ on $T \times X$ fitting in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{V}_\bullet) & \longrightarrow & \mathcal{V}'_0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(\mathcal{V}_\bullet) & \longrightarrow & \widetilde{\mathcal{V}}_0 & \longrightarrow & (h \times 1_X)^*(\mathcal{I}_1) & \longrightarrow & 0. \end{array} \tag{1.12}$$

Then by shrinking T, we have a morphism $f : T \to P^s$ with a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{V}_\bullet) & \longrightarrow & (f \times 1_X)^*(\widehat{\mathcal{V}}_0) & \longrightarrow & (f \times 1_X)^*(\widetilde{\mathcal{I}}_1) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & H^0(\mathcal{V}_\bullet) & \longrightarrow & \widetilde{\mathcal{V}}_0 & \longrightarrow & (f \times 1_X)^*(\widetilde{\mathcal{I}}_1) & \longrightarrow & 0. \end{array} \tag{1.13}$$

Therefore $(f \times 1_X)^*(\widehat{\mathcal{V}}_\bullet) \cong (\widetilde{\mathcal{V}}_0 \to W_1 \otimes \mathcal{O}_{T\times X}(-n_1))$ is quasi-isomorphic to $\mathcal{V}'_\bullet$. □

1.2. **Albanese map.** Let $\widehat{X}$ be the dual abelian variety of X and $\mathbf{P}$ the Poincaré line bundle on $\widehat{X} \times X$. Let $\mathfrak{a} : \mathbf{D}(X) \to \operatorname{Pic}(\widehat{X}) \times \operatorname{Pic}(X)$ be the morphism sending E to $(\det \Phi^{\mathbf{P}}_{X\to\widehat{X}}(E), \det(E))$. For a family of coherent sheaves $\mathcal{E}$ parametrized by a connected scheme T, we also have a morphism $\mathfrak{a} : T \to X \times \widehat{X}$ (up to translation).

We quote the following assertions from [Y2, Thm. 0.1, Lemma 4.3, Prop. 4.4].

Proposition 1.2. *Let v be a Mukai vector.*

(i) *Let $\mathcal{E}$ be a family of coherent sheaves on X with $v(\mathcal{E}_q) = v$ parametrized by a scheme Q. Assume that for any point $(x, y) \in X \times \widehat{X}$, $T^*_x(\mathcal{E}_q) \otimes \mathbf{P}_y \cong \mathcal{E}_{q'}$ for a point $q' \in Q$. Then $\dim \mathfrak{a}(Q) \geq 2$ and $\dim \mathfrak{a}(Q) = 4$ if $\langle v^2 \rangle > 0$.*

(ii) *If v is a primitive Mukai vector with $\langle v^2 \rangle > 0$. Then $\mathfrak{a} : M_H(v) \to \widehat{X} \times X$ is the Albanese map.*

In the case where $\langle v(E)^2 \rangle = 0$, we use Lemma 4.3 in [Y2] and the fact that $\phi_L = 0$ if and only if $c_1(L) = 0$.

2. Proof of Theorem 0.1

2.1. **Fourier-Mukai transform of a general stable sheaf.** Let Y be a moduli space of stable semi-homogeneous sheaves on X. Assume that there is a universal family $\mathbf{E}$ on $Y \times X$. Then we have a Fourier-Mukai transform $\Phi^{\mathbf{E}}_{Y\to X} : \mathbf{D}(Y) \to \mathbf{D}(X)$. If $\dim \mathbf{E}_y = 0$, $y \in Y$, then $\Phi^{\mathbf{E}}_{Y\to X}$ comes from an equivalence $\operatorname{Coh}(Y) \to \operatorname{Coh}(X)$. This case is easier to treat than other cases. In particular, the proof of Theorem 0.1 is reduced to the case treated in 2.3. Hence we assume that $\dim \mathbf{E}_y \geq 1$, $y \in Y$.

Theorem 2.1. *Let w be a primitive Mukai vector such that $\langle w^2 \rangle > 0$. If $\Phi^{\mathbf{E}}_{Y\to X}(E)$ is not a sheaf up to shift for all $E \in M_{H'}(w)$, then there is an integer k such that for a general $E \in M_{H'}(w)$, $\Phi^{\mathbf{E}}_{Y\to X}(E)[k]$ fits in an exact triangle*

$$A_0 \to \Phi^{\mathbf{E}}_{Y\to X}(E)[k] \to A_1[-1] \to A_0[1], \tag{2.1}$$

where $A_i, i = 0, 1$ are semi-homogeneous sheaves of $v(A_i) = n'_i v'_i$, $(n'_0 - 1)(n'_1 - 1) = 0$ and $\langle v'_0, v'_1 \rangle = -1$. In particular $\Phi_{\mathbf{E}}$ induces a birational map $M_{H'}(w) \cdots \to M_H(v)$, if $\operatorname{NS}(X) \cong \mathbb{Z}$ and $v \neq \pm(v'_0 - nv'_1)$ for all isotropic vectors v'_0, v'_1 with $\langle v'_0, v'_1 \rangle = -1$, where $v := \Psi^{\mathbf{E}}_{Y\to X}(w)$.

Proof. Let $Q(w)$ be the open subscheme of $\operatorname{Quot}^w_{\mathcal{O}_Y(-n)^{\oplus N}/Y}$ such that $M_{H'}(w)$ is the geometric quotient of $Q(w)$ by the action of $PGL(N)$. Let $\mathcal{E}$ be the universal family on $Q(w) \times Y$. Then for a point $q \in Q(w)$, we have

$$\begin{cases} H^0(\Phi^{\mathbf{E}}_{Y\to X}(\mathcal{E}_q)) = 0, & \mu(\mathcal{E}_q \otimes \mathbf{E}_x) \leq 0 \\ H^2(\Phi^{\mathbf{E}}_{Y\to X}(\mathcal{E}_q)) = 0, & \mu(\mathcal{E}_q \otimes \mathbf{E}_x) > 0, \end{cases} \tag{2.2}$$

where $x \in X$. Hence $\Phi^{\mathbf{E}}_{Y\to X}(\mathcal{E})[k]$ is a family of complexes represented by

$$\mathcal{V}_\bullet : \mathcal{V}_0 \to \mathcal{V}_1, \tag{2.3}$$

where $k = 1$ or $k = 0$. Assume that WIT does not hold for all $\mathcal{E}_q$. Let $Q(w)_0$ be the open subscheme such that $H^i(\mathcal{V}_\bullet)$ are flat over $Q(w)_0$. Let S_i be the bounded set of coherent sheaves $H^i(\mathcal{V}_\bullet)_q, q \in Q(w)_0$. We set $v_i := v(H^i(\mathcal{V}_\bullet)_q)$ and consider the family of complexes $\widehat{\mathcal{V}}_\bullet$ parametrized by P. Then for any point $q \in Q(w)_0$,

there is a neighborhood Q_q of q and a morphism $f_q : Q_q \to P$. We note that $Q(w)_0$ is $GL(N)$-invariant. We set $M_H(w)_0 := Q(w)_0/GL(N)$. By shrinking $Q(w)_0$ to an open subscheme, we may assume that the Harder-Narasimhan filtrations $0 \subset F_i^1 \subset F_i^2 \subset \cdots \subset F_i^{s_i} = H^i(\mathcal{V}_\bullet)_q, q \in Q(w)_0$ form a flat family of filtrations over $Q(w)_0$, that is, F_i^j/F_i^{j-1} form flat families of sheaves. We set $v_i^j := v(F_i^j/F_i^{j-1})$ and consider the locally closed subset Q_i' of Q_i such that

$$Q_i' = \left\{ W_i \otimes \mathcal{O}_X(-n_i) \to A_i \;\middle|\; \begin{matrix} \text{the Harder-Narasimhan filtration of } A_i \text{ is} \\ 0 \subset F_i^1 \subset F_i^2 \subset \cdots \subset F_i^{s_i} = A_i,\, v(F_i^j/F_i^{j-1}) = v_i^j \end{matrix} \right\} \tag{2.4}$$

(cf. Remark 2.1). Then we have a morphism

$$\begin{array}{rcccccc} \mathfrak{a}_i' : & Q_i' & \to & \prod_j \overline{M}_H(v_i^j) & \to & (X \times \widehat{X})^{s_i} \\ & A_i & \mapsto & \prod_j F_i^j/F_i^{j-1} & \mapsto & \prod_j \mathfrak{a}(F_i^j/F_i^{j-1}). \end{array} \tag{2.5}$$

By the proof of [Y2, Thm. 4.14], we get $\dim \mathfrak{a}_i'(Q_i') \geq 2s_i$. Indeed if n_i is sufficiently large, then we can show that the quotient stack $[Q_i'/GL(W_i)]$ is an affine bundle over $\prod_j \mathcal{M}_H(v_i^j)^{ss}$ (see [Y3, sect.2.2, in particular Prop. 2.5]). Combining this with Proposition 1.2, we get $\dim \mathfrak{a}_i'(Q_i') \geq 2s_i$. We set $P' := P \times_{(Q_0 \times Q_1)} (Q_0' \times Q_1')$. Then the image of $f_q : Q_q \to P$ is contained in P'. Let $\mathfrak{b} : P' \to Q_0' \times Q_1' \to (X \times \widehat{X})^{s_0+s_1}$ be the morphism defined by composing π with $\mathfrak{a}_0' \times \mathfrak{a}_1'$. Then $\dim \operatorname{im} \mathfrak{b} \geq 4$ and if the equality holds, then $s_0 = s_1 = 1$ and $\langle v_0^2 \rangle = \langle v_1^2 \rangle = 0$. Thus Q_i' are open subset of Q_i and $\mathcal{A}_i$ are families of semi-homogeneous sheaves.

Let P^s be the open subset of P' such that $\Phi_{X\to Y}^{\mathbf{E}^\vee}(\widehat{\mathcal{V}}_\bullet)[2-k]$ is a family of stable sheaves. Then we have a morphism $g : P^s \to M_H(w)$. Obviously $g \circ f_q : Q_q \to M_H(w)$ is the restriction of the quotient map ϖ. Combining with $\mathfrak{b}$, we have a morphism $Q_q \to P^s \to (X \times \widehat{X})^{s_0+s_1}$. This is the morphism determined by $\mathcal{E}_q$:

$$Q(w)_0 \ni q \mapsto (\mathfrak{a}_0'(H^0(\Phi_{Y\to X}^{\mathbf{E}}(\mathcal{E}_q)[k])), \mathfrak{a}_1'(H^1(\Phi_{Y\to X}^{\mathbf{E}}(\mathcal{E}_q)[k]))) \in (X \times \widehat{X})^{s_0} \times (X \times \widehat{X})^{s_1}. \tag{2.6}$$

Hence we have a morphism $\mathfrak{c} : M_H(v)_0 \to (X \times \widehat{X})^{s_0+s_1}$ such that $\mathfrak{c} \circ g = \mathfrak{b}$. Since $(X \times \widehat{X})^{s_0+s_1}$ is an abelian variety and $M_H(w)$ is smooth, $\mathfrak{c}$ extends to a morphism $M_H(v) \to (X \times \widehat{X})^{s_0+s_1}$. We also denote it by $\mathfrak{c}$.

On the other hand, $\mathfrak{a} : M_H(v) \to Y \times \widehat{Y}$ is the Albanese map of $M_H(v)$. Hence there is a morphism $a : Y \times \widehat{Y} \to (X \times \widehat{X})^{s_0+s_1}$ with $a \circ \mathfrak{a} = \mathfrak{c}$ and we have the following commutative diagram:

$$\begin{tikzcd} M_H(v) \arrow[d, "\mathfrak{a}"'] & P^s \arrow[l, "g"'] \arrow[d, "\mathfrak{b}"] \\ Y \times \widehat{Y} \arrow[r, "a"] & (X \times \widehat{X})^{s_0+s_1} \end{tikzcd}$$

Hence $\dim \operatorname{im} \mathfrak{b} \leq 4$, which implies that $H^j(\Phi_{Y\to X}^{\mathbf{E}}(\mathcal{E})[k])$, $j = 0, 1$ are families of semi-homogeneous sheaves. We set $v_i := n_i' v_i'$, where v_i' are primitive. Then $\langle v^2 \rangle = \langle (v_0 - v_1)^2 \rangle = -2n_0'n_1'\langle v_0', v_1' \rangle$. Hence $\langle v_0', v_1' \rangle < 0$. For a point $q \in Q(w)_0$, $V_\bullet : V_0 \to V_1$ denotes $(\mathcal{V}_\bullet)_q$. We set $A_i := H^i(V_\bullet)$. Then A_i are semi-homogeneous sheaves with $v(A_i) = v_i$. Since $\operatorname{Hom}_{\mathbf{D}(X)}(V_\bullet, V_\bullet) \cong \mathbb{C}$, $V_\bullet$ is not quasi-isomorphic to $A_0 \oplus A_1[1]$. Hence $\operatorname{Hom}_{\mathbf{D}(X)}(A_1[-1], A_0[1]) \neq 0$. Then $\operatorname{Ext}^2(A_1, A_0) = \operatorname{Hom}_{\mathbf{D}(X)}(A_1[-1], A_0[1]) \neq 0$ and $\operatorname{Ext}^1(A_1, A_0) = \operatorname{Hom}(A_1, A_0) = 0$ (see (3.8) and Remark 3.1 in Appendix). By Proposition 3.1, $\operatorname{Ext}^i((\mathcal{A}_1)_{q_1}, (\mathcal{A}_0)_{q_0}) = 0$, $i \neq 0$ for all $(q_0, q_1) \in Q_0' \times Q_1'$ and $\operatorname{Ext}^2_{p_{Q_0'\times Q_1'}}(\mathcal{A}_1, \mathcal{A}_0)$ is a locally free sheaf on $Q_0' \times Q_1'$ and all 2-extensions are parametrized by the associated vector bundle $\overline{P} \to Q_0' \times Q_1'$, where we also denote the pull-backs of $\mathcal{A}_i$ to $Q_0' \times Q_1' \times X$ by the same $\mathcal{A}_i$. $\overline{P}$ is a quotient bundle of P. We denote the image of P^s to $\overline{P}$ by $\overline{P}^s$. Then we have a morphism $\overline{g} : \overline{P}^s \to M_H(v)$ such that g is the composite $P^s \to \overline{P}^s \xrightarrow{\overline{g}} M_H(v)$. Since $\mathcal{A}_i$ are $GL(W_i)$-equivariant, $G := (GL(W_0) \times GL(W_1))/\mathbb{C}^\times$ acts on $\overline{P}$. By Lemma 3.2 in Appendix, G acts freely on $\overline{P}^s$ and the fiber of $\overline{g}$ is the G-orbit. By Corollary 3.6, $\dim Q_i' - \dim GL(W_i) = \dim \mathcal{M}_H(n_i'v_i')^{ss} = n_i'$, and hence

$$\begin{aligned} \dim \overline{g}(\overline{P}^s) &= \dim \operatorname{Ext}^2(A_1, A_0) + n_0' + n_1' + 1 \\ &= -n_0'n_1'\langle v_0', v_1' \rangle + n_0' + n_1' + 1. \end{aligned} \tag{2.7}$$

Then we get

$$\begin{aligned} \dim M_H(v) - \dim \overline{g}(\overline{P}^s) &= -2n_0'n_1'\langle v_0', v_1' \rangle + 2 - (-n_0'n_1'\langle v_0', v_1' \rangle + n_0' + n_1' + 1) \\ &= n_0'n_1'(-\langle v_0', v_1' \rangle - 1) + (n_0' - 1)(n_1' - 1), \end{aligned} \tag{2.8}$$

which implies that $\langle v_0', v_1' \rangle = -1$ and $(n_0' - 1)(n_1' - 1) = 0$. The last claim follows from [Y2, Cor. 4.15]. □

Remark 2.1. We note that g extends to a morphism from an open subset of P. Hence even if we do not know $\dim \operatorname{Alb}(M_{H'}(w))$, the closure of Q_i' should be a union of irreducible components of Q_i.

Remark 2.2. In the proof of Lemma 2.2 below, we shall see that $M_H(v)$ is birationally equivalent to $\widehat{Z} \times \mathrm{Hilb}_Z^{\langle v^2 \rangle/2}$ for an abelian surface Z.

2.2. **Reduction to the case where $V_\bullet$ is a sheaf up to shift.** If $\operatorname{rk} A_0 = \operatorname{rk} A_1 = 0$, then $c_1(A_0)$ and $c_1(A_1)$ are effective, and hence $\langle v(A_0), v(A_1) \rangle = (c_1(A_0), c_1(A_1)) \geq 0$. This is a contradiction. Since $\mathrm{Hom}(A_0, A_1) = \mathrm{Ext}^2(A_1, A_0)^\vee \neq 0$, we see that A_0 is locally free. We first show the following:

Lemma 2.2. *Keep notations as above. There is a Fourier-Mukai functor $\mathcal{F} : \mathbf{D}(X) \to \mathbf{D}(X)_{op}$ such that $\mathcal{F}(v) = v$ and one of the following three conditions holds*

(1) $\operatorname{rk}(H^0(\mathcal{F}(V_\bullet))) + \operatorname{rk} H^1(\mathcal{F}(V_\bullet))) < \operatorname{rk} H^0(V_\bullet) + \operatorname{rk} H^1(V_\bullet)$*, or*
(2) $\deg H^1(\mathcal{F}(V_\bullet)) < \deg H^1(V_\bullet)$ *if* $\operatorname{rk} H^1(V_\bullet) = 0$*, or*
(3) $H^1(\mathcal{F}(V_\bullet)) = 0$ *if* $H^1(V_\bullet)$ *is of 0-dimensional.*

Proof. (i) We first assume that $n'_1 = 1$. Since $\langle v'_0, v'_1 \rangle = -1$, $X_0 := M_H(v'_0)$ is a fine moduli space. Let $\mathbf{F}$ be the universal family of stable semi-homogeneous sheaves on $X_0 \times X$. Applying $\Phi_{X \to X_0}^{\mathbf{F}^\vee}$ to the exact triangle

$$A_0 \to V_\bullet \to A_1[-1] \to A_0[1], \tag{2.9}$$

we get an exact triangle

$$\Phi_{X \to X_0}^{\mathbf{F}^\vee}(A_0) \to \Phi_{X \to X_0}^{\mathbf{F}^\vee}(V_\bullet) \to \Phi_{X \to X_0}^{\mathbf{F}^\vee}(A_1)[-1] \to \Phi_{X \to X_0}^{\mathbf{F}^\vee}(A_0)[1]. \tag{2.10}$$

By Proposition 3.1, $L := \Phi_{X \to X_0}^{\mathbf{F}^\vee}(A_1)$ is a line bundle on X_0. We note that $G := \Phi_{X \to X_0}^{\mathbf{F}^\vee}(A_0)[2]$ is a 0-dimensional sheaf of length n'_0 on X_0. Hence (2.10) is

$$G[-1] \to \Phi_{X \to X_0}^{\mathbf{F}^\vee}(V_\bullet)[1] \to L \xrightarrow{f} G. \tag{2.11}$$

We can choose a general point $q \in Q(w)_0$ such that f is surjective. Then $G \cong \mathcal{O}_Z \otimes L$ for a 0-dimensional subscheme Z of n'_0-points and we get an exact sequence

$$0 \to H^1(\Phi_{X \to X_0}^{\mathbf{F}^\vee}(V_\bullet)) \to L \xrightarrow{f} \mathcal{O}_Z \otimes L \to 0. \tag{2.12}$$

Thus $\Phi_{X \to X_0}^{\mathbf{F}^\vee}(V_\bullet) = I_Z \otimes L[-1]$. By taking the dual, we have an exact triangle

$$\mathcal{O}_Z^\vee \to L^\vee \to (I_Z \otimes L)^\vee \to \mathcal{O}_Z^\vee[1]. \tag{2.13}$$

We note that $\mathcal{O}_Z^\vee = \mathcal{E}xt^2_{\mathcal{O}_{X_0}}(\mathcal{O}_Z, \mathcal{O}_{X_0})[-2] \cong \mathcal{O}_Z[-2]$, if Z consists of disjoint n'_1-points. We fix a line bundle L_0 on X_0 with $c_1(L_0) = c_1(L)$. For (2.13), by taking a tensor product $\otimes L_0^{\otimes 2}$ and applying $\Phi_{X_0 \to X}^{\mathbf{F}}$, we get an exact triangle

$$\Phi_{X_0 \to X}^{\mathbf{F}}(I_Z^\vee \otimes (L^\vee \otimes L_0^{\otimes 2}))[1] \to A_0 \xrightarrow{e'} B_1 \to \Phi_{X_0 \to X}^{\mathbf{F}}(I_Z^\vee \otimes (L^\vee \otimes L_0^{\otimes 2}))[2], \tag{2.14}$$

where $A_0 \cong H^2(\Phi_{X_0 \to X}^{\mathbf{F}}(\mathcal{O}_Z^\vee \otimes (L^\vee \otimes L_0^{\otimes 2})))$ and $B_1 := H^2(\Phi_{X_0 \to X}^{\mathbf{F}}(L^\vee \otimes L_0^{\otimes 2}))$ is a stable semi-homogeneous sheaf with $v(B_1) = v(A_1)$. We set $A'_0 := \ker e'$ and $A'_1 := \operatorname{coker} e'$. By shrinking $Q(w)_0$, we may assume that A'_i form flat families over $Q(w)_0$. Since $A_0 \to B_1$ is not trivial, we get the assertions.

(ii) We next assume that $n'_0 = 1$. In this case, we consider the Fourier-Mukai transform $\Phi_{X \to X_1}^{\mathbf{F}^\vee} : \mathbf{D}(X) \to \mathbf{D}(X_1)$, where $X_1 := M_H(v'_1)$ and $\mathbf{F}$ is the universal family on $X_1 \times X$. Then we have an exact triangle

$$L \to \Phi_{X \to X_1}^{\mathbf{F}^\vee}(V_\bullet)[2] \to \Phi_{X \to X_1}^{\mathbf{F}^\vee}(A_1[1]) \to L[1] \tag{2.15}$$

where $L := \Phi_{X \to X_1}^{\mathbf{F}^\vee}(A_0)[2]$ is a line bundle on X_0. For a general $q \in Q(w)_0$, we may assume that $\Phi_{X \to X_1}^{\mathbf{F}^\vee}(A_1) = \mathcal{O}_Z[2]$ for a subscheme of distinct n'_1-points Z on X_1. Then $(\mathcal{O}_Z[2])^\vee \cong \mathcal{O}_Z$. Hence by taking the dual of (2.15), we get an exact triangle

$$L^\vee \to \mathcal{O}_Z \to \Phi_{X \to X_1}^{\mathbf{F}^\vee}(V_\bullet)^\vee[-1] \to L^\vee[1]. \tag{2.16}$$

We fix a line bundle L_1 on X_1 with $c_1(L_1) = c_1(L)$. Since $B_0 := \Phi_{X \to X_1}^{\mathbf{F}}(L^\vee \otimes L_1^{\otimes 2})$ is a stable semi-homogeneous vector bundle with the Mukai vector v_1 and $\Phi_{X \to X_1}^{\mathbf{F}}(\mathcal{O}_Z) = A_1$, we have an exact triangle

$$B_0 \to A_1 \to \Phi_{X \to X_1}^{\mathbf{F}}(\Phi_{X \to X_1}^{\mathbf{F}^\vee}(V_\bullet)^\vee \otimes L_1^{\otimes 2})[-1] \to B_0[1], \tag{2.17}$$

which implies that the assertions holds. □

Applying Lemma 2.2 successively, we get a Fourier-Mukai functor $\mathcal{F} : \mathbf{D}(X) \to \mathbf{D}(X)$ or $\mathcal{F} : \mathbf{D}(X) \to \mathbf{D}(X)_{op}$ such that $\mathcal{F}(V_\bullet)$ is a sheaf with $v(\mathcal{F}(V_\bullet)) = v$.

2.3. **Proof of Theorem 0.1 (the case where $\Phi_{Y\to X}^{\mathbf{E}}(E)$ is a sheaf).** Replacing $V_\bullet$ by $\mathcal{F}(V_\bullet)$, we may assume that WIT_k holds for a general $\mathcal{E}_q$. Assume that $V := H^k(\Phi_{Y\to X}^{\mathbf{E}}(\mathcal{E}_q))$ is not stable. By [Y2, Thm. 4.14], V fits in an exact sequence

$$0 \to A_0 \to V \to A_1 \to 0, \tag{2.18}$$

where A_i are semi-homogeneous sheaves with the Mukai vector $n_i' v_i'$, $\langle v_0', v_1' \rangle = 1$ and $(n_0'-1)(n_1'-1) = 0$. We may assume that A_i are direct sum of distinct stable sheaves $A_{ij} \in M_H(v_i')$, $j = 1, 2, \dots, n_i'$. By using the following lemma, we shall replace the extension (2.18) by an extension in another direction.

Lemma 2.3. *Let V fits in an exact sequence*

$$0 \to A_0 \to V \to A_1 \to 0, \tag{2.19}$$

with $A_i = \oplus_j A_{ij}$, $A_{ij} \in M_H(v_i')$, $j = 1, 2, \dots, n_i'$ and $\langle v_0', v_1' \rangle = 1$. Then there is a Fourier-Mukai transform $\mathcal{F} : \mathbf{D}(X) \to \mathbf{D}(X)_{op}$ such that $\mathcal{F}(V)$ fits in an exact sequence

$$0 \to B_1 \to \mathcal{F}(V) \to B_0 \to 0, \tag{2.20}$$

where $B_i = \oplus_j B_{ij}$, $B_{ij} \in M_H(v_i')$, $j = 1, 2, \dots, n_i'$.

Proof. By the symmetry of the condition, we may assume that $n_1' = 1$. We set $X_0 := M_H(v_0')$ and $\mathbf{F}$ the universal family on $X_0 \times X$. Since $\chi(A_1, A_0) < 0$, IT_1 holds for A_1 and $L := H^1(\Phi_{X\to X_0}^{\mathbf{F}^\vee}(A_1))$ is a line bundle on X_0. We fix a line bundle L_0 with $c_1(L_0) = c_1(L)$. Then we see that $V' := \Phi_{X_0\to X}^{\mathbf{F}}(\Phi_{X\to X_0}^{\mathbf{F}^\vee}(V)^\vee \otimes L_1^{\otimes 2})$ is a sheaf and fits in an exact sequence

$$0 \to B_1 \to V' \to A_0 \to 0, \tag{2.21}$$

where $B_1 := \Phi_{X_0\to X}^{\mathbf{F}}(L^\vee \otimes L_0^{\otimes 2})[1] \in M_H(v_1)$. We set $B_0 := A_0$. Then the claim holds. □

We shall show that the instability is improved, under the operation $\mathcal{F}$ in Lemma 2.3. We only treat the case where $\operatorname{rk} V > 0$. The other cases are similar. For the exact sequence (2.18), by using Lemma 2.3, we replace V by $\mathcal{F}(V)$. Since (2.18) is the Harder-Narasimhan filtration, A_1 and hence B_1 is locally free. Assume that $V' := \mathcal{F}(V)$ is not stable for all point $q \in Q(w)$. Then a general V' fits in an exact sequence

$$0 \to A_0' \to V' \to A_1' \to 0, \tag{2.22}$$

where (1) $A_i' = \oplus_j A_{ij}', i = 0, 1$ are direct sum of distinct stable semi-homogeneous sheaves A_{ij}' with $v(A_{ij}') = v(A_{ik}')$ for all j, k, and (2) A_0' is a torsion sheaf, or V' is torsion free and $0 \subset A_0' \subset V'$ is the Harder-Narasimhan filtration of V'.

We shall divide the proof into three cases

(a) V is not torsion free.
(b) V is torsion free but not μ-semi-stable.
(c) V is μ-semi-stable, but not stable.

(a) Assume that V has a torsion. Then A_0 is the torsion submodule of V. Since V is simple, we see that $\deg A_0 > 0$. We show that the degree of the torsion submodule of V' is strictly smaller than that of V, that is, $\deg A_0' < \deg A_0$, if V' has a torsion. Assume that V' has a torsion. Then A_0' is the torsion submodule of V'. Since B_1 is locally free, $\varphi : A_0' \to B_0$ is injective. If $\deg A_0' = \deg B_0$, then φ is surjective in codimension 1. By using the locally freeness of B_1, we see that $V' \cong B_1 \oplus B_0$, which is a contradiction. Thus $\deg(A_0') < \deg B_0$.

(b) Assume that V is torsion free, but not μ-semi-stable. Then B_0 is also locally free. If $\mu(A_0') > \mu(V')$, then $A_0' \to B_0$ is not zero, which implies that $\mu(A_0') \le \mu(B_0)$. If $\mu(A_0') = \mu(B_0)$, then we also see that $A_0' \to B_0$ is injective, $n_0' = 1$ and A_0' is a direct summand of V'. Therefore $\mu(A_0') < \mu(B_0) = \mu(A_0)$. We can also see that $\mu(A_1') > \mu(A_1)$. Indeed since $\mu(A_0') \ge \mu(V') > \mu(B_1)$, $A_1' \to B_1$ is not zero. Then we have a non-trivial homomorphism $A_{1j}' \to B_1$ for a j and we see that $\mu(A_1') = \mu(A_{1j}') > \mu(A_1)$.

(c) If V is μ-semi-stable, i.e., $\mu(A_0) = \mu(A_1)$, then by the same argument, we see that $\chi(A_0')/\operatorname{rk} A_0' < \chi(A_0)/\operatorname{rk} A_0$ and $\chi(A_1')/\operatorname{rk} A_1' > \chi(A_1)/\operatorname{rk} A_1$. Therefore by applying Lemma 2.3 successively, we get a stable sheaf. Thus we complete the proof of Theorem 0.1. □

2.4. **In the case where Y is not fine.** In the notation in section 2.1, even if Y is not fine, there is a universal family as a $p_Y^*(\alpha^{-1})$-twisted sheaf for a suitable Cech 2-cocycle α of $\mathcal{O}_X^\times$. Then we have an equivalence

$$\Phi_{Y\to X}^{\mathbf{E}} : \mathbf{D}^\alpha(Y) \to \mathbf{D}(X), \tag{2.23}$$

where $\mathbf{D}^\alpha(Y) := \mathbf{D}(\mathrm{Coh}^\alpha(X))$ is the bounded derived category of coherent α-twisted sheaves. Let $M_{H'}^\alpha(w)$ be the moduli space of stable α-twisted stable sheaves E with $v(E) = w$. If $\dim \operatorname{Alb} M_H(w) = 4$, then Theorem 0.1 also holds for this case. By a similar method as in [Y5], we can show that $\dim \operatorname{Alb}(M_{H'}^\alpha(w)) = 4$, if

$\langle w^2 \rangle > 0$ (cf. 3.4 in Appendix). Here we treat one example by another argument based on [Y4]. In the same way as in [Y4, Prop. 3.14], we see that for a stable α-twisted sheaf E of rank 0, $\Phi^{\mathbf{E}}_{Y \to X}(E(nH'))$ is stable for $n \gg 0$. In particular, we have an isomorphism $M^{\alpha}_{H'}(we^{nH'}) \cong M_H(v')$, where $v(E) = w$ and $v(\Phi^{\alpha}_{Y \to X}(E(nH'))) = v'$. In particular $\mathrm{Alb}(M^{\alpha}_{H'}(w)) \cong X \times \widehat{X}$. Then by the same proof as in Theorem 0.1, we see that $M^{\alpha}_{H'}(w)$ is birationally equivalent to $M_H(v)$. Since the support map $M^{\alpha}_{H'}(w) \to \mathrm{Hilb}^{c_1(w)}_Y$ $(E \mapsto \mathrm{Div}(E))$ is a Lagrangian fibration, we get the following:

Proposition 2.4. *Assume that there is an primitive isotropic vector v_0 such that v_0 is algebraic and $\langle v, v_0 \rangle = 0$. Then $M_H(v)$ is birationally equivalent to a holomorphic symplectic manifold with a Lagrangian fibration.*

Corollary 2.5. *The Albanese fiber $K_H(v)$ is birationally equivalent to an irreducible symplectic manifold with a Lagrangian fibration if and only if $\mathrm{Pic}(K_H(v))$ has an isotropic element with respect to the Beauville form.*

For related results on Lagrangian fibrations on irreducible symplectic manifolds, see [G],[M],[S] and references therein.

3. Appendix

3.1. **Semi-homogeneous sheaves.** The following assertions are well-known (cf. [Mu1], [O]).

Proposition 3.1. *Let E and F be semi-homogeneous sheaves.*

(i) *Assume that E and F are locally free sheaves.*
  (a) *If $\langle v(E), v(F) \rangle > 0$, then* $\mathrm{Hom}(E, F) = \mathrm{Ext}^2(E, F) = 0$.
  (b) *If $\langle v(E), v(F) \rangle < 0$, then $\mu(E) \neq \mu(F)$,* $\mathrm{Ext}^1(E, F) = 0$ *and*

$$\begin{cases} \mathrm{Hom}(E,F) = 0, & \mu(E) > \mu(F) \\ \mathrm{Ext}^2(E,F) = 0, & \mu(F) > \mu(E). \end{cases} \tag{3.1}$$

(ii) *Assume that E is locally free and F is a torsion sheaf.*
  (a) *If $\langle v(E), v(F) \rangle > 0$, then* $\mathrm{Hom}(E, F) = \mathrm{Ext}^2(E, F) = 0$.
  (b) *If $\langle v(E), v(F) \rangle < 0$, then* $\mathrm{Ext}^1(E, F) = \mathrm{Ext}^2(E, F) = 0$.

(iii) *Assume that E and F are torsion sheaves. Then $\langle v(E), v(F) \rangle \geq 0$. If $\langle v(E), v(F) \rangle > 0$, then* $\mathrm{Hom}(E, F) = \mathrm{Ext}^2(E, F) = 0$.

This is equivalent to the fact that the Fourier-Mukai transform of a semi-homogeneous sheaf is a sheaf up to shift.

Proof. We only prove (i). Indeed the proof of (ii) and (iii) are reduced to (i) via a suitable Fourier-Mukai transform. We note that $E^{\vee} \otimes F$ is semi-homogeneous. There is a filtration $\subset F_1 \subset F_2 \subset \cdots \subset F_s = E^{\vee} \otimes F$ such that $E_i = F_i/F_{i=1}$, $1 \leq i \leq s$ are simple semi-homogeneous vector bundles with $c_1(E_i)/\operatorname{rk} E_i = c_1(E^{\vee} \otimes F)/(\operatorname{rk} E \operatorname{rk} F) = c_1(F)/\operatorname{rk} F - c_1(E)/\operatorname{rk} E$. Since $\chi(E_i)/\operatorname{rk} E_i = (c_1(E_i)/\operatorname{rk} E_i)^2/2 = \chi(E, F)/\operatorname{rk} E \operatorname{rk} F$, it is sufficient to prove the claim for $E = \mathcal{O}_X$ and F is a simple semi-homogeneous vector bundle. Then there is an isogeny $\pi : Y \to X$ and a line bundle L on Y such that $\pi_*(L) = F$. Hence $\mathrm{Ext}^i(E, F) = H^i(X, F) = H^i(Y, L)$. In particular, $\chi(E, F) = \chi(L) = (c_1(L)^2)/2$. If $\chi(L) < 0$, then $H^i(Y, L) = 0$ for $i \neq 1$. Thus (a) holds. Assume that $\chi(L) > 0$. Since $\pi_*(c_1(L)) = c_1(F)$, $(c_1(L), \pi^*(H)) = (c_1(F), H)$. If $\mu(F) = 0$, then the Hodge index theorem implies that $(c_1(L)^2) \leq 0$, which is a contradiction. Therefore $\mu(F) \neq 0 = \mu(E)$. The other claims also follow from $(c_1(L), \pi^*(H)) = (c_1(F), H)$. □

3.2. **2-extensions.** We collect some elementary facts on 2-extensions. We have a natural map

$$\Xi : \mathrm{Ext}^2(A_1, A_0) \to Ob(\mathbf{D}(X))/(\text{quasi-isom.}) \tag{3.2}$$

by sending a 2-extension class

$$0 \to A_0 \to V_0 \to V_1 \to A_1 \to 0 \tag{3.3}$$

to the complex $V_\bullet : V_0 \to V_1$. We want to study the fiber of Ξ. We take a resolution

$$0 \to E_{-2} \to E_{-1} \to E_0 \to A_1 \to 0 \tag{3.4}$$

such that $H^j(X, E_i^{\vee} \otimes A_0) = 0$ for $i = 0, -1$, $j > 0$. Then we also have $H^j(X, E_{-2}^{\vee} \otimes A_0) = 0$ for $j > 0$. Hence $\mathrm{Ext}^2(A_1, A_0) \cong \mathrm{Hom}(E_{-2}, A_0)/\mathrm{im}(\mathrm{Hom}(E_{-1}, A_0))$ and for a representative $\varphi \in \mathrm{Hom}(E_{-2}, A_0)$, $\Xi([\varphi])$ is the cone $V_\bullet$ defined by

$$\varphi : E_\bullet[-2] \to A_0. \tag{3.5}$$

For two exact triangles,

$$A_0 \to V^i_\bullet \to A_1[-1] \to A_0[1], \ i = 1, 2, \tag{3.6}$$

we have an exact and commutative diagram:

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(X)}(A_1[-1], A_0) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet^1, A_0) & \longrightarrow & \mathrm{Hom}(A_0, A_0) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(X)}(A_1[-1], V_\bullet^2) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet^1, V_\bullet^2) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(X)}(A_0, V_\bullet^2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(A_1[-1], A_1[-1]) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet^1, A_1[-1]) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}(X)}(A_0, A_1[-1]) = 0.
\end{array}
\tag{3.7}
$$

Hence we have an exact sequence

$$
0 \to \mathrm{Hom}_{\mathbf{D}(X)}(A_1[-1], A_0) \xrightarrow{i} \mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet^1, V_\bullet^2) \xrightarrow{r} \mathrm{Hom}(A_0, A_0) \oplus \mathrm{Hom}(A_1[-1], A_1[-1]). \tag{3.8}
$$

We take a quasi-isomorphism $(V_\bullet^1)' \to V_\bullet^1$ such that $\mathrm{Ext}^j((V_1^1)', V_i^2) = \mathrm{Ext}^j((V_1^1)', A_0) = 0$ for $j > 0, i = 0, 1$ and $(V_i^1)' = 0$ for $i \neq 0, 1$. Then $\mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet^1, V_\bullet^2)$ is the cohomology group of the complex

$$
\mathrm{Hom}((V_1^1)', V_0^2) \to \mathrm{Hom}((V_0^1)', V_0^2) \oplus \mathrm{Hom}((V_1^1)', V_1^2) \to \mathrm{Hom}((V_0^1)', V_1^2). \tag{3.9}
$$

Then $\varphi \in \mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet^1, V_\bullet^2)$ induces an exact and commutative diagram:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A_0 & \longrightarrow & (V_0^1)' & \xrightarrow{\phi'} & (V_1^1)' & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle\varphi_0}\downarrow & & \downarrow{\scriptstyle\varphi_1} & & \downarrow & & \\
0 & \longrightarrow & A_0 & \longrightarrow & V_0^2 & \xrightarrow{\phi} & V_1^2 & \longrightarrow & A_1 & \longrightarrow & 0
\end{array}
\tag{3.10}
$$

Conversely this diagram gives an element $\phi \in \mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet^1, V_\bullet^2)$. For

$$
\varphi \in \mathrm{Hom}_{\mathbf{D}(X)}(A_1[-1], A_0) = \mathrm{Hom}(\mathrm{im}\,\phi', A_0)/\mathrm{Hom}((V_1^1)', A_0), \tag{3.11}
$$

$i(\varphi)$ is represented by $(\varphi \circ \phi', 0) \in \mathrm{Hom}((V_0^1)', V_0^2) \oplus \mathrm{Hom}((V_1^1)', V_1^2)$.

We have an action of $\mathrm{Aut}(A_0) \times \mathrm{Aut}(A_1)$ on $\mathrm{Ext}^2(A_1, A_0)$:

$$
\begin{array}{rccc}
(g_0, g_1): & \mathrm{Ext}^2(A_1, A_0) & \to & \mathrm{Ext}^2(A_1, A_0) \\
 & e & \mapsto & g_0 \cup e \cup g_1^{-1}.
\end{array}
\tag{3.12}
$$

It is easy to see that the following lemma holds.

Lemma 3.2.

$$
\begin{aligned}
\Xi^{-1}(\Xi(e)) &= (\mathrm{Aut}(A_0) \times \mathrm{Aut}(A_1))e, \\
r(\mathrm{Aut}_{\mathbf{D}(X)}(V_\bullet)) &= \{(g_0, g_1) | g_0 \cup e \cup g_1^{-1} = e\}
\end{aligned}
\tag{3.13}
$$

for $e \in \mathrm{Ext}^2(A_1, A_0)$ *with* $V_\bullet = \Xi(e)$. *In particular,* $GL(W_0) \times GL(W_1)/\mathbb{C}^\times$ *acts freely on the open subscheme of* $\overline{P}$ *parametrizing simple complexes* $V_\bullet$, *where* $\overline{P}$ *is the scheme in the proof of Theorem 2.1.*

Remark 3.1. $\mathrm{Hom}(A_1, A_0) \cong \mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet, V_\bullet[-1])$. If $V_\bullet$ is the Fourier-Mukai transform of a sheaf E, then it is 0.

3.2.1. *Some remarks on the endomorphisms of complexes.* We shall show that for a complex $\widehat{V}_\bullet$ as in section 1, $\mathrm{Hom}_{\mathbf{D}(X)}(\widehat{V}_\bullet, \widehat{V}_\bullet)$ is represented by a morphism $\widehat{V}_\bullet \to \widehat{V}_\bullet$ up to homotopy.

Lemma 3.3. *Let* $V_\bullet : V_0 \to V_1$ *be a complex. Let* $V_\bullet' : \cdots \to V_{-1}' \to V_0' \to V_1' \to 0$ *be a complex and* $f : V_\bullet' \to V_\bullet$ *a quasi-isomorphism. Then* $\mathrm{Hom}_{\mathbf{K}(X)}(V_\bullet, V_\bullet) \to \mathrm{Hom}_{\mathbf{K}(X)}(V_\bullet', V_\bullet)$ *is injective, where* $\mathbf{K}(X)$ *is the homotopy category of complexes. In particular,* $\mathrm{Hom}_{\mathbf{K}(X)}(V_\bullet, V_\bullet) \to \mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet, V_\bullet)$ *is injective.*

Proof. Let $W_\bullet$ be the cone of $f : V_\bullet' \to V_\bullet$. We have an exact sequence $\cdots \to V_0' \to V_0 \oplus V_1' \to V_1 \to 0$. Then it is easy to see $\mathrm{Hom}_{\mathbf{K}(X)}(W_\bullet, V_\bullet) = 0$. So the claim follows. □

Lemma 3.4. *Let* $V_\bullet : V_0 \xrightarrow{d} V_1$ *be a complex. We set* $A_i := H^i(V_\bullet)$. *Assume that* $\mathrm{Hom}(V_1, V_1) \cong \mathrm{Hom}(V_1, A_1)$ *and* $\mathrm{Ext}^1(V_1, A_0) = 0$. *Then* $\mathrm{Hom}_{\mathbf{D}(X)}(V_\bullet, V_\bullet) \cong \mathrm{Hom}_{\mathbf{K}(X)}(V_\bullet, V_\bullet)$.

Proof. We take a quasi-isomorphism $f : V'_\bullet \to V_\bullet$ such that $H^i(V'_\bullet) = 0$, $i \neq 0, 1$, $\operatorname{Ext}^j(V'_1, V_i) = \operatorname{Ext}^j(V'_1, A_0) = 0$ for $j > 0$ and $V'_1 \to V_1$ is surjective. Then $\operatorname{Hom}_{\mathbf{K}(X)}(V'_\bullet, V_\bullet) \cong \operatorname{Hom}_{\mathbf{D}(X)}(V_\bullet, V_\bullet)$. We note that there is an exact and commutative diagram:

$$\tag{3.14}\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \ker f_0 \arrow[r] \arrow[d] & \ker f_1 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_0 \arrow[r] \arrow[d, "H^0(f)"'] & V'_0 \arrow[r, "d'"] \arrow[d, "f_0"'] & V'_1 \arrow[r] \arrow[d, "f_1"] & A_1 \arrow[r] \arrow[d, "H^1(f)"] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[d] & V_0 \arrow[r, "d"] \arrow[d] & V_1 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

It is sufficient to show the surjectivity of $\operatorname{Hom}_{\mathbf{K}(X)}(V_\bullet, V_\bullet) \to \operatorname{Hom}_{\mathbf{K}(X)}(V'_\bullet, V_\bullet)$. Let $\phi : V'_\bullet \to V_\bullet$ be a morphism. Since f is a quasi-isomorphsm, we have a morphism $a : A_1 \to A_1$ such that $a \circ H^1(f) = H^1(\phi)$. By our assumption, there is a morphism $g : V_1 \to V_1$ with a commutative diagram

$$\tag{3.15}\begin{tikzcd}
V_1 \arrow[r] \arrow[d, "g"'] & A_1 \arrow[d, "a"] \\
V_1 \arrow[r] & A_1
\end{tikzcd}$$

Since $\operatorname{Ext}^1(V'_1, A_0) = 0$ and the image of $\phi_1 - g \circ f_1$ is contained in $d(V_0)$, we have a morphism $\lambda : V'_1 \to V_0$ such that $d \circ \lambda = \phi_1 - g \circ f_1$. Replacing ϕ_1 by $\phi_1 - d \circ \lambda$ and ϕ_0 by $\phi_0 - \lambda \circ d'$, we may assume that $\phi_1 = g \circ f_1$. By the above diagram, we have $d \circ \phi_0(\ker f_0) = 0$, which implies that $\phi_{0|\ker f_0} \in \operatorname{Hom}(\ker f_0, A_0)$. Since $\operatorname{Ext}^1(V_1, A_0) = 0$, there is a $\lambda' : V'_1 \to A_0$ such that $(\phi_0 - \lambda')_{|\ker f_0} = 0$. So replacing ϕ_0 by $\phi_0 - \lambda'$, we have a morphism $\phi'_0 : V_0 \to V_0$ with $\phi_0 = \phi'_0 \circ f_0$. Thus (ϕ'_0, g) gives a desired morphism $V_\bullet \to V_\bullet$. □

Remark 3.2. For a complex $V_\bullet$, we can find a quasi-isomorphism $\widehat{V}_\bullet \to V_\bullet$ as in section 1 such that $\widehat{V}_\bullet$ satisfies the assumptions of this lemma.

Remark 3.3. By our assumption, we see that $\operatorname{Hom}(V_1, d(V_0)) = 0$. Then the kernel of r in (3.8) consists of (ϕ'_0, g) such that $g = 0$ and ϕ'_0 comes from a morphism $d(V_0) \to A_1$. Thus

$$\tag{3.16} \ker r = \operatorname{Hom}(d(V_0), A_0)/\operatorname{Hom}(V_1, A_0) = \operatorname{Hom}_{\mathbf{D}(X)}(A_1[-1], A_0).$$

This is compatible with (3.8).

3.3. **Twisted sheaves.** Let $X = \cup_i U_i$ be an analytic open covering of X and $\alpha = \{\alpha_{ijk}\}$ a Cech 2-cocycle of $\mathcal{O}_X^\times$ representing a torsion element $[\alpha] \in H^2(X, \mathcal{O}_X^\times)$. Let $\mathcal{M}_H^\alpha(v)^{ss}$ be the moduli stack of semi-stable α-twisted sheaves of Mukai vector v.

Lemma 3.5. *We set $v := (0, 0, n)$. Then $\dim \mathcal{M}_H^\alpha(v)^{ss} = n$.*

Proof. We fix an α-twsited vector bundle G of rank r on X. Let E be a 0-dimensional α-twisted sheaf of length n. Then $\operatorname{Hom}(G, E) \otimes G \to E$ is surjective. We set $Q := \operatorname{Quot}^v_{G^{\oplus rn}/X}$. Then $\mathcal{M}_H^\alpha(v)^{ss}$ is the quotient stack of Q by the natural action of $GL(rn)$: $\mathcal{M}_H^\alpha(v)^{ss} = [Q/GL(rn)]$. We claim that $\dim Q = (r^2 n + 1)n$. Then $\dim \mathcal{M}_H^\alpha(v)^{ss} = \dim Q - \dim GL(rn) = (r^2 n + 1)n - (rn)^2 = n$ and we get our lemma. So we shall prove the claim. We have a natural morphism $\phi : Q \to \overline{M}_H^\alpha(v)$. Since $M_H^\alpha(0, 0, 1) \cong X$, there is a bijective morphism $\psi : S^n X \to \overline{M}_H^\alpha(v)$. In order to prove the claim, it is sufficient to show $\dim \phi^{-1}(\psi(\sum_{i=1}^s n_i P_i)) = \sum_i (r^2 n n_i - 1)$, where $P_1, ..., P_s$ are distinct points of X. We set $Z := \operatorname{Spec}\mathbb{C}[[x, y]]$. Since the punctual quot-scheme $\operatorname{Quot}^m_{\mathcal{O}_Z^{\oplus l}/Z}$ is of dimension $lm - 1$ (cf. [Y1] or [N-Y, Cor. 3.7]), we get our claim. □

Corollary 3.6. *Let v_0 be a primitive Mukai vector with $\langle v_0^2 \rangle = 0$ and $\operatorname{rk} v_0 > 0$. Then $\dim \mathcal{M}_H(nv_0)^{ss} = n$.*

Proof. For a sufficiently large m, every semi-stable sheaf F with $v(F) = nv_0$ is a quotient of $\operatorname{Hom}(\mathcal{O}_X(-m), F) \otimes \mathcal{O}_X(-m)$:

$$\tag{3.17} 0 \to \ker \psi \to \operatorname{Hom}(\mathcal{O}_X(-m), F) \otimes \mathcal{O}_X(-m) \xrightarrow{\psi} F \to 0.$$

We set $Y := M_H(v_0)$. Let $\mathbf{E}$ be the universal family on $X \times Y$ as a $p_Y^*(\alpha)$-twisted sheaf, where α is a suitable $\mathcal{O}_Y^\times$ coefficient 2-cocycle and p_Y is the projection. Since $m \gg 0$, we have an exact sequence

$$\tag{3.18} 0 \to \Phi_{X \to Y}^{\mathbf{E}^\vee}(\ker \psi)[2] \to \operatorname{Hom}(G, E) \otimes G \to E \to 0,$$

where $G := \Phi_{X \to Y}^{\mathbf{E}^\vee}(\mathcal{O}_X(-m))[2]$ and $E := \Phi_{X \to Y}^{\mathbf{E}^\vee}(F)[2]$. Hence we have an isomorphism $\mathcal{M}_H(nv_0)^{ss} \cong [Q/GL(rn)]$, where $r = \operatorname{rk} G$ and Q is the scheme in Lemma 3.5. Therefore we get our claim. □

3.4. **Weight 1 Hodge structure.** Let α be a Cech 2-cocycle of $\mathcal{O}_X^\times$ representing a r-torsion element of $H^2(X, \mathcal{O}_X^\times)$. We have a homomorphism

$$H^2(X, \mathbb{Z}/r\mathbb{Z}) \to H^2(X, \mathcal{O}_X^\times) \tag{3.19}$$

whose image is the set of r-torsion elements. We take a representative $\xi \in H^2(X, \mathbb{Z})$ such that $[\xi \mod r] \in H^2(X, \mathbb{Z}/r\mathbb{Z})$ maps to $[\alpha]$.

Definition 3.1. We define a weight 1 Hodge structure on $H^{odd}(X, \mathbb{Z})$ as

$$\begin{aligned} H^{1,0}(H^*(X,\mathbb{Z}) \otimes \mathbb{C}) :=& e^{\xi/r}(H^{1,0}(X) \oplus H^{2,1}(X)) \\ H^{0,1}(H^*(X,\mathbb{Z}) \otimes \mathbb{C}) :=& e^{\xi/r}(H^{0,1}(X) \oplus H^{1,2}(X)). \end{aligned} \tag{3.20}$$

We denote this Hodge structure by $(H^{odd}(X, \mathbb{Z}), -\frac{\xi}{r})$.

Let v be a primitive Mukai vector with $\langle v^2 \rangle > 0$. Then by a similar argument as in [Y5], we have an isomorphism $H^{odd}(X, \mathbb{Z}) \cong H^1(M_H^\alpha(v), \mathbb{Z})$ preserving the Hodge structure. Indeed, we use the surjectivity of the period map (the period map is a double covering). In particular, we get $\dim \operatorname{Alb}(M_H^\alpha(v)) = 4$.

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References

[G] Gulbrandsen, Martin, G., *Lagrangian fibrations on generalized Kummer varieties,* math.AG/0510145

[M] Markushevich, D., *Rational Lagrangian fibrations on punctual Hilbert schemes of K3 surfaces,* math.AG/0509346

[Mu1] Mukai, S., *Semi-homogeneous vector bundles on an Abelian variety,* J. Math. Kyoto Univ. **18** (1978), 239–272

[Mu2] Mukai, S., *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves,* Nagoya Math. J., **81** (1981), 153–175

[N-Y] Nakajima, H., Yoshioka, K., *Lectures on instanton counting*, Algebraic structures and moduli spaces, 31–101, CRM Proc. Lecture Notes, 38, Amer. Math. Soc., Providence, RI, 2004.

[O] Orlov, D., *Derived categories of coherent sheaves on abelian varieties and equivalences between them,* alg-geom/9712017, Izvestia RAN, Ser.Mat., **66**, (2002) 131-158

[S] Sawon, J., *Lagrangian fibrations on Hilbert schemes of points on K3 surfaces,* math.AG/0509224

[Y1] Yoshioka, K., *The Betti numbers of the moduli space of stable sheaves of rank 2 on $\mathbb{P}^2$,* J. reine angew. Math. **453** (1994), 193–220

[Y2] Yoshioka, K., *Moduli spaces of stable sheaves on abelian surfaces,* Math. Ann. **321** (2001), 817–884, math.AG/0009001

[Y3] Yoshioka, K., *Twisted stability and Fourier-Mukai transform II,* Manuscripta Math. **110** (2003), 433–465

[Y4] Yoshioka, K., *Stability and the Fourier-Mukai transform II,* preprint (see sections 3, 4 of math.AG/0112267)

[Y5] Yoshioka, K., *Moduli of twisted sheaves on a projective variety,* math.AG/0411538, Adv. Stud. Pure Math. to appear

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